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Operatorial quantization of the Born–Infeld Skyrmion model and hidden symmetries

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Abstract

The $SU(2)$ collective coordinates expansion of the Born–Infeld Skyrmion Lagrangian is performed. The classical Hamiltonian is computed from this special Lagrangian in an approximative way: it is derived from the expansion of this non-polynomial Lagrangian up to second-order variable in the collective coordinates. This second-class constrained model is quantized by the Dirac Hamiltonian method and symplectic formalism. Although it is not expected to find symmetries on second-class systems, a hidden symmetry is disclosed by formulating the Born–Infeld Skyrmion model as a gauge theory. To this end we developed a new constraint conversion technique based on the symplectic formalism. Finally, a discussion on the role played by the hidden symmetry on the computation of the energy spectrum is presented.

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1. Introduction

The presence of symmetries in a constrained dynamical model reveals important physical contents of a given system. In particular, we can cite the energy spectrum that is an observable quantity invariant under gauge transformations. In view of this, some authors [1] have proposed some processes to convert second-class systems to first-class ones. In this paper, we are interested in investigating this subject by employing a new technique based on the symplectic procedure, called the symplectic gauge-invariant method [2]. To clarify our proposal we will quantize a Skyrme-like model, discussed in this paper.

The Skyrme model [3] is an effective field theory for baryons and their interactions. These hadronic particles are described from soliton solutions in the non-linear sigma model. Normally, in this Lagrangian, it is necessary to add the Skyrme term to stabilize the soliton solutions. In principle, the Skyrme term is arbitrary and there is no concrete reason to fix it through a particular choice [4]. Its importance resides in the fact that, maybe, it is the simplest possible quartic derivative term that we need to insert in the Hamiltonian in order to obtain soliton solutions[1]. However, it is possible to avoid this ambiguity by adopting a nonconventional Lagrangian also based on a non-linear sigma model, given by

$$L = -\frac{F_\pi}{16}\int \mathrm{d}^3r\,[\mathrm{Tr}\,\partial_\mu U\partial^\mu U^+]^{\frac{3}{2}} \tag{1}$$

where F_π is the pion decay constant and U is a $SU(2)$ matrix. This model, based on the ideas of Born–Infeld electrodynamics [6], was proposed by Deser *et al* [7]. The existence of soliton solutions might be observed by applying Derrick's theorem in the static Hamiltonian derived from the Lagrangian (1).

The main goal of this paper is to quantize the Born–Infeld Skyrmion model through the symplectic gauge-invariant method and then to display the role played by the hidden symmetry on the computation of the energy spectrum. To this end, this model is expanded in terms of the collective rotational coordinates and, subsequently, formulated as a gauge-invariant theory. Implementing the semi-classical approach after the usual collective canonical quantization, the spin and isospin modes are obtained, producing quantum corrections to the baryon's properties [8]. This process reduces the $SU(2)$ Skyrme model to a nonrelativistic particle constrained over a $\mathcal{S}^3$ sphere, a well known second-class problem [9, 10]. Afterwards, the $SU(2)$ Skyrme model, expanded in terms of the collective rotational coordinates [8], is quantized via the Dirac Hamiltonian method [11] and symplectic formalism [12, 13], which allows us to compute the field dependent Dirac brackets among the physical coordinates, assumed to be commutators at the quantum level. We observe that, when we keep the non-causality sector of the soliton solution influencing the physical values as minimum as possible, the commutators obtained are the same as in the Skyrme model. At this level, problems involving operator ordering ambiguities [14, 15] arise, which can be avoided just by formulating this model as a gauge-invariant theory. At this stage we unveil a hidden symmetry of the model, which is an unexpected result since this nonlinear model is originally a second-class model.

For the sake of self-consistency, this paper was organized as follows. In section 2, we propose the Born–Infeld Skyrmion model, obtaining the classical Hamiltonian in an approximative way from the expansion of the non-polynomial Lagrangian up to second-order variable in the collective coordinates, since we take into account some causality arguments. In section 3, the second-class model will be quantized via Dirac and symplectic methods, where we demonstrate that the computed Dirac brackets are the same as those obtained for the Skyrme model [16, 17]. In section 4, the Born–Infeld Skyrme model will be reformulated as a gauge theory via a symplectic gauge-invariant method. In this section we will also investigate the hidden symmetry lying on the original phase-space coordinates. In order to corroborate the previous results, we investigate in section 5 the hidden symmetry via the gauge unfixing Hamiltonian method [18]. In section 6, the role played by the symmetry on the computation of the energy spectrum will be explored. The last section is dedicated to the discussion of the physical meaning of our findings together with our final comments and conclusions. In appendices 1 and 2, we will present brief reviews of the new constraint conversion procedure, namely the symplectic gauge-invariant formalism [2], and the gauge unfixing Hamiltonian formalism, respectively.

2. The Born–Infeld Skyrmion model

The dynamic system will be given by performing the $SU(2)$ collective semi-classical expansion [8]. Substituting $U(\vec{r}, t)$ by $A(t)U(\vec{r})A^+(t)$ in (1), where A is a $SU(2)$ matrix,

[1] According to the Derrick scale theorem [5].

we obtain

$$L = -F_\pi \int \mathrm{d}^3 r \left[m - I \ \mathrm{Tr}(\partial_0 A \partial_0 A^{-1})\right]^{\frac{3}{2}} \tag{2}$$

where m and I, identified as being the soliton mass and the inertia moment [8], respectively, are functionals written in terms of the chiral angle $F(r)$, which satisfies the topological boundary conditions, $F(0) = \pi$ and $F(\infty) = 0$. Here, we use the hedgehog ansatz for U, i.e., $U = \exp(\mathrm{i}\tau \cdot \hat{r} F(r))$. The $SU(2)$ matrix A can be written as $A = a_0 + \mathrm{i} a_i \tau_i$, where τ_i are the Pauli matrices, leading to the constraint

$$\sum_{i=0}^{i=3} a_i a_i = 1. \tag{3}$$

The Lagrangian (2) can be written as a function of the a_i as

$$L = -\frac{F_\pi}{16} \int \mathrm{d}^3 r [m - 2I \, \dot{a}_i \dot{a}_i]^{\frac{3}{2}}. \tag{4}$$

From equation (4) we can obtain the conjugate momenta, given by

$$\pi_i = \frac{\partial L}{\partial \dot{a}_i} = \frac{3F_\pi}{16} \, \dot{a}_i \int \mathrm{d}^3 r \, I [m - 2I \, \dot{a}_k \dot{a}_k]^{\frac{1}{2}}. \tag{5}$$

The algebraic expression for the Hamiltonian is obtained by applying the Legendre transformation, $H = \pi_i \dot{a}_i - L$. However, in some situations, due to the momenta expression given in equation (5), it is not possible to write the conjugate Hamiltonian corresponding to the Born–Infeld Skyrmion Lagrangian in terms of π_i and a_i. An alternative procedure is to expand the original Lagrangian (4) in collective coordinates. Thus, considering the binomial expansion variable[2] $\frac{I}{m} \dot{a}_i \dot{a}_i$, the Lagrangian sum is given by

$$L = -M + A(\dot{a}_i \dot{a}_i) - B(\dot{a}_i \dot{a}_i{}^2) + \cdots \tag{6}$$

where $M = \frac{F_\pi}{16} \int \mathrm{d}^3 r \, m^{\frac{3}{2}}$, $A = \frac{3F_\pi}{16} \int \mathrm{d}^3 r \ I \sqrt{m}$, $B = \frac{3F_\pi}{32} \int \mathrm{d}^3 r \ \frac{I^2}{\sqrt{m}}$, etc. In this step we would like to give a physical argument that justifies this procedure. Even though not being a relativistic invariant model, we hope that the experimental results can be reproducible with good accuracy when the soliton velocity is much smaller than the speed of light. From the relation given in [19], $A^+ \partial_0 A = \mathrm{i}/2 \sum_{k=1}^{k=3} \tau_k \omega_k$, where ω_k is the uniform soliton angular velocity, it is possible to show that $\mathrm{Tr}[\partial_0 A \partial_0 A^+] = 2\dot{a}_i \dot{a}_i = \omega^2/2$. If we require that the soliton rotates with velocity smaller than c, then $\omega r \ll 1$, leading to $\dot{a}_i \dot{a}_i = \frac{\omega^2}{4} \ll 1$, and consequently $\dot{a}_i \dot{a}_i \ll 1$ for all space. Thus, these results explain our procedure.

In this manner, the Hamiltonian is obtained by using the Legendre transformation

$$\begin{aligned} H &= \pi_i \dot{a}_i - L \\ &= M + A(\dot{a}_i \dot{a}_i) - 3B(\dot{a}_i \dot{a}_i)^2 + \cdots. \end{aligned} \tag{7}$$

Obtaining the canonical momenta from equation (6), then writing the Lagrangian as $L = \pi_i \dot{a}_i - H$, and comparing with the expansion of the Lagrangian (6), it is possible to derive the expression of the Hamiltonian (7) as

$$H = M + \alpha \, \pi_i \pi_i + \beta \, (\pi_i \pi_i)^2 + \cdots \tag{8}$$

with $\alpha = \frac{1}{4A}$ and $\beta = \frac{B}{16A^4}$. We will truncate the expression (8) in the second-order variable[3], and we will use this approximate Hamiltonian to perform the quantization.

[2] In the context of semi-classical expansion, it is expected that the product of $\dot{a}_i \dot{a}_i$ by the expression $\frac{I}{m}$ given by the Euler–Lagrange equation does not considerably modify this result.

[3] Due to equation (5) together with the fact that $\dot{a}_i \dot{a}_i \ll 1$, we expect that terms like $(\pi_i \pi_i)^3$ or higher order degrees do not alter our conclusion about the commutators of the quantum Born–Infeld Skyrmion.

3. Operatorial quantization at the second-class level

In this section the reduced Born–Infeld Skyrmion model will be quantized using the Dirac method [11] and the symplectic formalism [12, 13]. In order to apply the Dirac second-class Hamiltonian method, we need to look for secondary constraints, which can be calculated from the following Hamiltonian:

$$H_T = M + \alpha\, \pi_i \pi_i + \beta\, (\pi_i \pi_i)^2 + \lambda_1 (a_i a_i - 1) \tag{9}$$

where λ_1 is a Lagrangian multiplier. Using the iterative Dirac formalism we get the following second-class constraints:

$$\phi_1 = a_i a_i - 1 \approx 0 \tag{10}$$

$$\phi_2 = a_i \pi_i \approx 0. \tag{11}$$

After straightforward computations, the Dirac brackets among the phase spaces variables are obtained as

$$\begin{aligned} \{a_i, a_j\}^* &= 0 \\ \{a_i, \pi_j\}^* &= \delta_{ij} - a_i a_j \\ \{\pi_i, \pi_j\}^* &= a_j \pi_i - a_i \pi_j . \end{aligned} \tag{12}$$

Through the well known canonical quantization rule $\{\ ,\ \}^* \to -\mathrm{i}[\ ,\ ]$, we get the commutators

$$\begin{aligned} [a_i, a_j] &= 0 \\ [a_i, \pi_j] &= -\mathrm{i}(\delta_{ij} - a_i a_j) \\ [\pi_i, \pi_j] &= -\mathrm{i}(a_j \pi_i - a_i \pi_j). \end{aligned} \tag{13}$$

These results show that the quantum commutators of the reduced Born–Infeld Skyrmion model are equal to the Skyrme model [16, 17] when the Lagrangian is expanded up to the second-order term of the collective coordinates. This completes the Dirac quantization process.

To implement the symplectic quantization procedure [13], let us consider the zeroth-iterative first-order Lagrangian:

$$L^{(0)} = \pi_i \dot{a}_i - V^{(0)} \tag{14}$$

where the potential $V^{(0)}$ is

$$V^{(0)} = M + \alpha \pi_i \pi_i + \beta (\pi_i \pi_i)^2 + \lambda (a_i a_i - 1) \tag{15}$$

with the enlarged symplectic variables given by $\xi_\alpha^{(0)} = (a_j, \pi_j, \lambda)$. The symplectic tensor is

$$f^{(0)} = \begin{pmatrix} 0 & -\delta_{ij} & 0 \\ \delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{16}$$

where the elements of rows and columns follow the order: a_i, π_i, λ. The matrix above is obviously singular, then it has a zero-mode that generates the following constraint:

$$\Omega^{(1)} = a_i a_i - 1 \approx 0 \tag{17}$$

where the potential $V^{(0)}$ is given by equation (15). Taking the time derivative of this constraint and introducing the result into the previous Lagrangian by means of a Lagrange multiplier ρ, we get a new Lagrangian $L^{(1)}$:

$$L^{(1)} = (\pi_i + \rho a_i)\dot{a}_i - V^{(1)} \tag{18}$$

where

$$V^{(1)} = M + \alpha \pi_i \pi_i + \beta (\pi_i \pi_i)^2. \tag{19}$$

The matrix $f^{(1)}$ is then

$$f^{(1)} = \begin{pmatrix} 0 & -\delta_{ij} & -a_i \\ \delta_{ij} & 0 & 0 \\ a_i & 0 & 0 \end{pmatrix} \tag{20}$$

where rows and columns follow the order: a_i, π_i, ρ. The matrix $f^{(1)}$ is singular so it has a zero-mode, given by

$$v^{(1)} = \begin{pmatrix} 0 \\ a_i \\ -1 \end{pmatrix} \tag{21}$$

that produces the constraint

$$\Omega^{(2)} = a_i \pi_i \approx 0. \tag{22}$$

Here we must mention that these constraints, equations (17) and (22), derived by the symplectic procedure are the same as those obtained when the Dirac formalism is used. Following the iterative symplectic process, we get the new Lagrangian $L^{(2)}$, given by

$$L^{(2)} = (\pi_i + \rho a_i + \eta \pi_i)\dot{a}_i + \eta a_i \dot{\pi}_i - V^{(2)} \tag{23}$$

where $V^{(2)} = V^{(1)}$. The new enlarged symplectic variables are $\xi_\alpha^{(2)} = (a_j, \pi_j, \rho, \eta)$, where ρ and η are Lagrange multipliers. The corresponding symplectic matrix $f^{(2)}$ is

$$f^{(2)} = \begin{pmatrix} 0 & -\delta_{ij} & -a_i & -\pi_i \\ \delta_{ij} & 0 & 0 & -a_i \\ a_i & 0 & 0 & 0 \\ \pi_i & a_i & 0 & 0 \end{pmatrix}. \tag{24}$$

The matrix $f^{(2)}$ is not singular and it is identified as the symplectic tensor of the constrained theory. The inverse of $f^{(2)}$ gives the same Dirac brackets among the physical coordinates given in equation (12).

4. Gauging the Born–Infeld Skyrmion model

The Born–Infeld Skyrme model is a non-invariant model with field dependent Dirac brackets among the phase-space variables. Due to this, the quantization of the model is affected by operator ordering ambiguity. To overcome this problem at the commutator level, the model will be reformulated as a gauge-invariant model. In this section, we will use the symplectic gauge-invariant method proposed in section 4. To implement this scheme, the second-order Lagrangian that governs the dynamics of the Born–Infeld Skyrmion model is reduced to its first-order form and an extra term $G(a_i, \pi_i, \theta)$ is introduced into the first-iterative Lagrangian, namely

$$L^{(1)} = \pi_i \dot{a}_i + (a_i^2 - 1)\dot{\eta} - V^{(1)} \tag{25}$$

where $-\lambda \to \dot{\eta}$ and with $V^{(1)}$ as

$$V^{(1)} = V^{(0)}|_{(a^2-1=0)} = M + \frac{1}{4A}\pi_i^2 + \frac{B}{16A^4}\pi_i^4 - G(a_i, \pi_i, \theta). \tag{26}$$

The symplectic variables are $\xi_\alpha^{(1)} = (a_i, \pi_i, \eta, \theta)$ and the extra term, given by

$$G(a_i, \pi_i, \theta) = \sum_n^\infty \mathcal{G}^{(n)}(a_i, \pi_i, \theta) \tag{27}$$

satisfies the boundary condition:

$$G(a_i, \pi_i, \theta = 0) = \mathcal{G}^{(0)}(a_i, \pi_i, \theta = 0) = 0. \tag{28}$$

The corresponding symplectic matrix, computed as

$$f^{(1)} = \begin{pmatrix} 0 & -\delta_{ij} & 2a_i & 0 \\ \delta_{ij} & 0 & 0 & 0 \\ -2a_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{29}$$

is singular and has a zero-mode:

$$\nu^{(1)} = (\,0 \quad a_i \quad \tfrac{1}{2} \quad 1\,)\,. \tag{30}$$

Following the prescription of the symplectic gauge-invariant formalism, giving in appendix 1, the gauge-invariant Lagrangian obtained after the fourth iteration is

$$L^{(1)} = \pi_i \dot{a}_i + (a_i^2 - 1)\dot{\eta} - V_{(4)}^{(1)} \tag{31}$$

where the symplectic potential $V_{(4)}^{(1)}$ is identified as being the invariant Hamiltonian:

$$\begin{aligned} H = M + \frac{1}{4A}\pi_i^2 + \frac{B}{16A^4}\pi_i^4 - \left(\frac{1}{2A} + \frac{B}{4A^4}\pi_i^2\right)(a_i\pi_i)\theta \\ + \left(a_i^2 + a_i^2\frac{B}{A^3}(a_i\pi_i)^2 + \frac{B}{2A^3}a^2\pi^2\right)\frac{\theta^2}{4A} - \frac{B}{4A^4}(a_i\pi_i)a^2\theta^3 + \frac{B}{16A^4}a^4\theta^4. \end{aligned} \tag{32}$$

To complete our gauge-invariant reformulation, the infinitesimal gauge transformation is also computed. In agreement with the symplectic method, the zero-mode $\tilde{\nu}^{(1)}$ is the generator of the infinitesimal gauge transformation ($\delta\mathcal{O} = \varepsilon\tilde{\nu}^{(1)}$):

$$\begin{aligned} \delta a_i &= 0 \\ \delta\pi_i &= \varepsilon a_i \\ \delta\lambda &= \frac{\dot{\varepsilon}}{2} \\ \delta\theta &= \varepsilon \end{aligned} \tag{33}$$

where ε is an infinitesimal time-dependent parameter.

At this stage, some gauge fixing schemes will be implemented following the symplectic method that allows us to reveal a new and remarkable result. First, we require that $\chi_1 = \theta = 0$ (unitary gauge) that reduces the gauge- invariant model to the original model with the same Dirac brackets among the phase-space variables (a_i, π_i). The other one is

$$\chi_2 = \lambda = 0. \tag{34}$$

With this gauge we have another non-invariant description for the nonlinear model with canonical Dirac brackets. In fact, the first-order Lagrangian becomes

$$L^{(1)} = \pi_i \dot{a}_i + \lambda\dot{\rho} - V_{(4)}^{(1)} \tag{35}$$

where the symplectic variables are $\xi_\alpha^{(1)} = (a_i, \pi_i, \lambda, \rho, \theta)$ and $V_{(4)}^{(1)}$ is

$$\begin{aligned} V_{(4)}^{(1)} = V_{(4)}^{(0)}|_{\lambda=0} = M + \frac{1}{4A}\pi_i^2 + \frac{B}{16A^4}\pi_i^4 - \left(\frac{1}{2A} + \frac{B}{4A^4}\pi_i^2\right)(a_i\pi_i)\theta \\ + \left(a^2 + a^2\frac{B}{A^3}(a_i\pi_i)^2 + \frac{B}{2A^3}a^2\pi^2\right)\frac{\theta^2}{4A} - \frac{B}{4A^4}(a_i\pi_i)a^2\theta^3 + \frac{B}{16A^4}a_i^4\theta^4. \end{aligned} \tag{36}$$

The corresponding symplectic matrix, computed as

$$f^{(1)} = \begin{pmatrix} 0 & -\delta_{ij} & 0 & 0 & 0 \\ \delta_{ij} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{37}$$

is singular and has a zero-mode that produces a new constraint:

$$\begin{aligned} \Omega_2 = \frac{1}{2A}(a_i\pi_i) + \frac{3B}{4A^4}(a_i\pi_i)a^2\theta^2 - \frac{B}{4A^4}a^4\theta^3 + \frac{B}{4A^4}\pi_i^2(a_i\pi_i) \\ -\frac{1}{2A}a_i^2\theta - \frac{B}{2A^4}(a_i\pi_i)^2\theta - \frac{B}{4A^4}\pi^2a^2\theta. \end{aligned} \tag{38}$$

With the introduction of this constraint into the first-order Lagrangian ($L^{(2)}$), we have

$$L^{(2)} = \pi_i\dot{a}_i + \lambda\dot{\rho} + \Omega_2\dot{\beta} - V_{(4)}^{(2)} \tag{39}$$

where $V_{(4)}^{(2)} = V_{(4)}^{(1)}|_{\Omega_2=0}$. After that, a nonsingular symplectic matrix is set up and the Dirac brackets among the phase-space variables are identified as

$$\begin{aligned} \{a_i, a_j\}^* &= 0 \\ \{a_i, \pi_j\}^* &= \delta_{ij} \\ \{\pi_i, \pi_j\}^* &= 0. \end{aligned} \tag{40}$$

From $\Omega_2 = 0$, the θ variable can be determined as

$$\theta = \frac{1}{a^2}\left(a_i\pi_i + \frac{B}{2A^3}\pi_i^2(a_i\pi_i)\right). \tag{41}$$

Bringing back this result into the symplectic potential $V_{(4)}^{(2)} = V_{(4)}^{(1)}|_{\Omega_2=0}$ and collecting terms up to π_i^4, we obtain the following Hamiltonian:

$$H = V^{(2)} = M + \frac{1}{4A}\pi_i M_{ij}\pi_j + \frac{B}{16A^4}(\pi_i M_{ij}\pi_j)^2 \tag{42}$$

with the singular matrix M_{ij} defined as

$$M_{ij} = \delta_{ij} - a_i a_j. \tag{43}$$

This Hamiltonian will be used to perform the computation of the energy spectrum. At this stage it is important to notice that the non-invariant model has a hidden symmetry that could not be detected by the symplectic method, due to the non-existence of a gauge generator. In spite of this, the Hamiltonian (42) is invariant under the gauge infinitesimal transformations (33), because the matrix M_{ij} has an eigenvector with eigenvalue null:

$$a_i M_{ij} = 0. \tag{44}$$

In the next section, the hidden symmetry will be investigated using the gauge unfixing method.

5. The gauge-invariant Born–Infeld Skyrmion model

In this section the hidden symmetry which underlies the Born–Infeld Skyrmion model will be disclosed using the gauge unfixing Hamiltonian method [18], reviewed in the appendix. This model has a set of second-class constraints, given in (10) and (11), that produces the nonvanishing Poisson bracket:

$$C = \{\phi_1, \phi_2\} = 2a_i a_i = 2. \tag{45}$$

To obtain the first-class Hamiltonian in a systematic way we follow closely the procedure described in [20]. Initially, the set of constraints are redefined as

$$\begin{aligned} \xi &= C^{(-1)}\phi_1 = \tfrac{1}{2}a_i a_i - \tfrac{1}{2} \\ \psi &= \phi_2 \end{aligned} \tag{46}$$

that generates the canonical Poisson bracket:

$$\{\xi, \psi\} = 1. \tag{47}$$

Subsequently, the Lagrange multipliers u_a with $a = 1, 2$ which appears in the total Hamiltonian:

$$H = H_c + u_1\xi + u_2\psi \tag{48}$$

with

$$H_c = M + \alpha\pi_i\pi_i + \beta(\pi_i\pi_i)^2 \tag{49}$$

are determined as

$$\begin{aligned} u_1 &= -2\alpha\pi_i^2 - 4\beta(\pi_i^2)^2 \\ u_2 &= -2\alpha a_i\pi_i - 4\beta(a_i\pi_i)\pi_i^2 \end{aligned} \tag{50}$$

just imposing that the constraints have no time evolution. To obtain the first-class system, we maintain only ξ as a gauge generator. At first, $\{\xi, H\} \neq 0$, i.e., ξ and H, in principle, do not satisfy a first-class algebra. Thus, the first-class Hamiltonian can be given by the following formula [20]:

$$\tilde{H} = H - \psi\{\xi, H\} + \frac{1}{2!}\psi^2\{\xi, \{\xi, H\}\} - \frac{1}{3!}\psi^3\{\xi, \{\xi, \{\xi, H\}\}\} + \cdots \tag{51}$$

with

$$H = M + \alpha\pi_i\pi_i + \beta(\pi_i\pi_i)^2 - 2\alpha(a_i\pi_i)^2 + 4\beta(a_i\pi_i)^2(\pi_j\pi_j) \tag{52}$$

satisfying the first-class algebra:

$$\{\xi, \tilde{H}\} = 0. \tag{53}$$

At this point, we start to compute each term of the Hamiltonian (51). The first one is

$$\{\xi, H\} = -2\alpha(a_i\pi_i) + 12\beta(a_i\pi_i)(\pi_i\pi_i) + 8\beta(a_i\pi_i)^3 \tag{54}$$

while the second and the remaining ones are given by

$$\begin{aligned} \{\xi, \{\xi, H\}\} &= -2\alpha + 12\beta(\pi_i\pi_i) + 48\beta(a_i\pi_i)^2 \\ \{\xi, \{\xi, \{\xi, H\}\}\} &= 120\beta(a_i\pi_i) \\ \{\xi, \{\xi, \{\xi, \{\xi, H\}\}\}\} &= 120\beta. \end{aligned} \tag{55}$$

The gauge-invariant Hamiltonian is obtained as

$$\begin{aligned} \tilde{H} &= M + \alpha(\pi_i\pi_i) + \beta(\pi_i\pi_i)^2 - 2\beta(a_i\pi_i)^2(\pi_j\pi_j) - \alpha(a_i\pi_i)^2 + \beta(a_i\pi_i)^4 \\ &= M + \frac{1}{4A}\pi_i M_{ij}\pi_j + \frac{B}{16A^4}(\pi_i M_{ij}\pi_j)^2 \end{aligned} \tag{56}$$

with the singular matrix M_{ij} given in equation (43).

It is easy to show that the gauge-invariant Hamiltonian satisfies the noninvolutive algebra:

$$\{\xi, \tilde{H}\} = 0. \tag{57}$$

Due to this, the constraint ξ is the gauge symmetry generator of the infinitesimal transformation:

$$\begin{aligned} \delta a_i &= \{a_i, \xi\} = 0 \\ \delta\pi_i &= \{\pi_i, \xi\} = -\varepsilon a_i \end{aligned} \tag{58}$$

with ε as an infinitesimal time-dependent parameter. Note that the Hamiltonian (56) is invariant under this infinitesimal gauge transformation because a_i are eigenvectors of the phase-space metric M_{ij} with null eigenvalues ($a_i M_{ij} = 0$).

6. The energy spectrum

In this section, we will derive the energy levels. Normally, these results were employed to obtain the baryon's physical properties [8]. In this context, our perturbative approach plays an important role on the computation of the energy spectrum since the quartic term presenting in the Hamiltonian (8) leads to an extra term.

In the second-class formalism the energy spectrum is obtained calculating the mean value of the quantum Hamiltonian, which is

$$E = \langle\psi|\tilde{H}|\psi\rangle \tag{59}$$

where $\tilde{H} = M+\alpha\,\pi_i\pi_i+\beta\,(\pi_i\pi_i)^2$. $\tilde{H}$ is the quantum version of the second-class Hamiltonian, equation (8). The eigenvectors of the quantum Hamiltonian $\tilde{H}$ are $|\psi\rangle = \frac{1}{N(l)}(a_1 + \mathrm{i}a_2)^l = |\mathrm{polynomial}\rangle$. These wave functions are also eigenvectors of the spin and isospin operators, written in [8] as $J_k = \frac{1}{2}(a_0\pi_k - a_k\pi_0 - \epsilon_{klm}a_i\pi_m)$ and $I_k = \frac{1}{2}(a_k\pi_0 - a_0\pi_k - \epsilon_{klm}a_i\pi_m)$. The expression for π_i, satisfying the commutation relations, equations (13), is given by

$$\pi_i = \frac{1}{\mathrm{i}}(\partial_i - a_i a_j\partial_j). \tag{60}$$

The algebraic expression for π_i presents operator ordering problems. A possible choice, following the prescription of Weyl ordering [21] (symmetrization procedure) is given by

$$\begin{aligned}[p_i]_{\mathrm{sym}} &= \frac{1}{6\mathrm{i}}(6\partial_i - a_i a_j\partial_i - a_i\partial_j a_j - a_j a_i\partial_i - a_j\partial_j a_i - \partial_j a_i a_j - \partial_j a_j a_i)\\ &= \frac{1}{\mathrm{i}}\left(\partial_i - a_i a_j\partial_j - \frac{5}{2}a_i\right).\end{aligned} \tag{61}$$

Consequently, $\pi_j\pi_j$ symmetrized can be written as

$$[\pi_j\pi_j]_{\mathrm{sym}} = -\partial_j\partial_j + \tfrac{1}{2}\big(\mathrm{OpOp} + 2\mathrm{Op} + \tfrac{5}{4}\big) \tag{62}$$

where Op is defined as $\mathrm{Op} \equiv a_i\partial_i$. The symmetrized second-class Hamiltonian operator is

$$[\tilde{H}]_{\mathrm{sym}} = M + \alpha\big[-\partial_j\partial_j + \tfrac{1}{2}\big(\mathrm{OpOp} + 2\mathrm{Op} + \tfrac{5}{4}\big)\big] + \beta\big[[-\partial_j\partial_j + \tfrac{1}{2}\big(\mathrm{OpOp} + 2\mathrm{Op} + \tfrac{5}{4}\big)]\big]^2. \tag{63}$$

Substitution of the expression (63) in the mean value, (59), leads to the energy levels[4]

$$E_l =_{\mathrm{phys}} \langle\psi|\tilde{H}|\psi\rangle_{\mathrm{phys}} = M + \alpha\big[l(l+2) + \tfrac{5}{4}\big] + \beta\big[l(l+2) + \tfrac{5}{4}\big]^2. \tag{64}$$

Notice that these energy levels have a quartic extra term, indicating some modifications on the calculation of the physical parameters, previously obtained in the context of the $SU(2)$ Skyrme model [8]. Furthermore, we remark that the adopted ordering scheme produces a constant value on the energy levels formula. It is an important subject since different ordering schemes can lead to distinct physical results, as pointed out in [15, 17].

In the first-class scenario the quantum Hamiltonian is

$$\tilde{H} = M + \alpha(\pi_i\pi_i) + \beta(\pi_i\pi_i)^2 - 2\beta(a_i\pi_i)^2(\pi_j\pi_j) - \alpha(a_i\pi_i)^2 + \beta(a_i\pi_i)^4. \tag{65}$$

$\tilde{H}$ is the quantum version of the first-class Hamiltonian, equations (42). The quantization is performed, following the prescription of the Dirac method [11], imposing that the physical wave functions are annihilated by the first-class operator constraint:

$$\phi_1|\psi\rangle_{\mathrm{phys}} = 0 \tag{66}$$

[4] Note that the eigenvalues of the operator Op are defined by the following equation: $\mathrm{Op}|\mathrm{polynomial}\rangle = l|\mathrm{polynomial}\rangle$.

where ϕ_1 is

$$\phi_1 = a_i a_i - 1. \tag{67}$$

The physical states that satisfy (66) are

$$|\psi\rangle_{\text{phys}} = \frac{1}{V}\,\delta(a_i a_i - 1)\,|\text{polynomial}\rangle \tag{68}$$

where V is the normalization factor. Thus, in order to obtain the spectrum of the theory, we take the scalar product, ${}_{\text{phys}}\langle\psi|\tilde{H}|\psi\rangle_{\text{phys}}$, that is the mean value of the first-class Hamiltonian. We begin by calculating the scalar product:

$${}_{\text{phys}}\langle\psi|\tilde{H}|\psi\rangle_{\text{phys}} = \langle\text{polynomial}|\;\frac{1}{V^2}\int \mathrm{d}a_i\;\delta(a^i a^i - 1)\;\tilde{H}\,\delta(a_i a_i - 1)\;|\text{polynomial}\rangle \tag{69}$$

where $\tilde{H}$ is defined in equations (65). Note that, due to $\delta(a_i a_i - 1)$ in (69), the scalar product can be simplified. Then, integrating over a_i, we obtain

$$\begin{aligned}{}_{\text{phys}}\langle\psi|\tilde{H}|\psi\rangle_{\text{phys}} = \langle\text{polynomial}|M + \alpha(\pi_i\pi_i) + \beta(\pi_i\pi_i)^2 - 2\beta(a_i\pi_i)^2(\pi_j\pi_j) - \alpha(a_i\pi_i)^2 \\ +\beta(a_i\pi_i)^4|\text{polynomial}\rangle.\end{aligned} \tag{70}$$

Here we would like to comment that the regularization delta function squared $\delta(a_i a_i - 1)^2$ is performed using the delta relation, $(2\pi)^2\delta(0) = \lim_{k\to 0}\int \mathrm{d}^2x\,\mathrm{e}^{\mathrm{i}k\cdot x} = \int \mathrm{d}^2x = V$. In this manner, the parameter V is used as the normalization factor. The Hamiltonian operator inside the kets, equation (70), can be rewritten as

$${}_{\text{phys}}\langle\psi|\tilde{H}|\psi\rangle_{\text{phys}} = \langle\text{polynomial}|M + \alpha[p_k \cdot p_k] + \beta[p_k \cdot p_k]^2|\text{polynomial}\rangle \tag{71}$$

where $p_k = \pi_k - a_k(a_j\pi_j)$. The operators π_k describe a free particle. Then, the p_k operators are identical to the canonical momenta obtained for the second-class theory. Consequently, the algebraic expression for the quantum Hamiltonian inside the scalar product (71) is the same obtained in a second-class theory, equation (63), naturally leading to the same energy levels, equation (64). This important result shows the equivalence between the second-class collective coordinates Skyrme model and its first-class version.

7. Conclusions

The Born–Infeld Skyrmion Lagrangian has a nonconventional structure that allows us to stabilize the soliton solutions without adding higher derivative order terms. However, due to the nonanalytical structure of a Born–Infeld Skyrmion Lagrangian, a perturbative treatment becomes necessary. The expansion of the non-polynomial Born–Infeld Lagrangian in terms of dynamic variables is possible if we pay attention to the problem of breaking the relativistic invariance in the collective coordinates expansion. The contributions to the physical parameters due to the non-causal soliton solution have no physical relevance if the chiral angle $F(r)$ satisfies the relation, $\lim_{r\to\infty} F(r) = 0$, together with the fact that we impose that the soliton angular velocity be small, i.e., $\omega \ll 1$. In view of this, the perturbative expression for the Hamiltonian could be truncated at the quartic-order term in the canonical momenta.

To obtain the quantum structure for the Born–Infeld Skyrmion model, the Dirac Hamiltonian method and the Lagrangian symplectic formalism were used. In these contexts, we verified that all constraints are second class and the symplectic matrix is nonsingular, showing that it is not expected to find symmetries. Afterward, the quantum commutators of the model were computed. These results are the same ones obtained for the conventional $SU(2)$ Skyrme model. In spite of this, the energy levels with a quartic correction term, together with an operator ordering scheme, can change the baryons static properties, previously obtained for the usual $SU(2)$ Skyrme model.

In order to disclose the hidden symmetry, two different approaches were used: the symplectic gauge-invariant scheme and the unfixing Hamiltonian formalism. The usual directions [22–24] point out the enlargement of the phase space with WZ variables, and consequently the symmetries arise. In our work, those symmetries are revealed only on the original phase space, where the quantum structure was obtained using the Dirac first-class procedure. In this scenario, the energy levels were computed, reproducing the spectrum of the original second-class system. Our findings point out the consistency of those first-class conversion procedures and propose the gauge-invariant version for the Born–Infeld Skyrmion model, dynamically equivalent to the usual $SU(2)$ Skyrme model.

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Appendix 1. The symplectic gauge-invariant method

There are several schemes to reformulate non-invariant models as gauge-invariant theories in the literature. However, recently, some constraint conversion formalisms, based on the Dirac method [11], were developed using Faddeev's idea of phase-space extension with the introduction of auxiliary variables [22]. Among them, the Batalin–Fradkin–Fradkina–Tyutin (BFFT) [23] and the iterative [24] methods were powerful enough to be successfully applied to a large number of important physical models. Although these techniques share the same conceptual basis [22] and treat constrained systems following the Dirac process [11], the implementation of the constraint conversion methods are different. Historically, both BFFT and the iterative methods were applied in linear systems, such as chiral gauge theories [24,25], in order to eliminate the gauge anomaly that hampers the quantization process. In spite of the great success achieved by these methods, some ambiguities which appear in the constraint conversion process make these iterative methods a hard task to implement [26]. It happens because these formalisms are based on the Dirac method. In this section, we reformulate non-invariant systems as gauge-invariant theories using a new technique which is not affected by those ambiguity problems. This technique follows the Faddeev suggestion [22] and is set up on a contemporary framework to handle non-invariant model, namely the symplectic formalism [12,13].

In order to systematize the symplectic gauge-invariant formalism, a general non-invariant mechanical model whose dynamics is governed by a Lagrangian $\mathcal{L}(a_i, \dot{a}_i, t)$(with $i = 1, 2, \ldots, N$) is considered, where a_i and $\dot{a}_i$ are the space and velocity variables, respectively. Notice that this consideration does not lead to the loss of generality or of physical content. Following the symplectic method, the first-order Lagrangian, written in terms of the symplectic variables $\xi_\alpha^{(0)}(a_i, p_i)$ (with $\alpha = 1, 2, \ldots, 2N$), is

$$\mathcal{L}^{(0)} = A_\alpha^{(0)} \dot{\xi}_\alpha^{(0)} - V^{(0)} \tag{72}$$

where $A_\alpha^{(0)}$ is the one-form canonical momenta, (0) indicates that it is the zeroth-iterative Lagrangian and $V^{(0)}$ is the symplectic potential. After that, the symplectic tensor, defined as

$$f_{\alpha\beta}^{(0)} = \frac{\partial A_\beta^{(0)}}{\partial \xi_\alpha^{(0)}} - \frac{\partial A_\alpha^{(0)}}{\partial \xi_\beta^{(0)}} \tag{73}$$

is computed. Since this symplectic matrix is singular, it has a zero-mode ($\nu^{(0)}$) that generates a new constraint when contracted with the gradient of potential, namely

$$\Omega^{(0)} = \nu_\alpha^{(0)} \frac{\partial V^{(0)}}{\partial \xi_\alpha^{(0)}}. \tag{74}$$

Through a Lagrange multiplier η, this constraint is introduced into the zeroth-iterative Lagrangian (72), generating the next one:

$$\begin{aligned}\mathcal{L}^{(1)} &= A_\alpha^{(0)} \dot{\xi}_\alpha^{(0)} - V^{(0)} + \dot{\eta}\Omega^{(0)} \\ &= A_\alpha^{(1)} \dot{\xi}_\alpha^{(1)} - V^{(1)}\end{aligned} \tag{75}$$

where

$$\begin{aligned}V^{(1)} &= V^{(0)}|_{\Omega^{(0)}=0} \\ \xi_\alpha^{(1)} &= (\xi_\alpha^{(0)}, \eta) \\ A_\alpha^{(1)} &= A_\alpha^{(0)} + \eta \frac{\partial \Omega^{(0)}}{\partial \xi_\alpha^{(0)}}.\end{aligned} \tag{76}$$

The first-iterative symplectic tensor is computed and, since this tensor is nonsingular, the iterative process stops and the Dirac brackets among the phase-space variables are obtained from the inverse matrix. In contrast, if the tensor is singular, a new constraint arises and the iterative process goes on.

After this brief review, the symplectic gauge formalism will be systematized. It starts after the first iteration with the introduction of an extra term dependent on the original and Wess–Zumino (WZ) variable, $G(a_i, p_i, \theta)$, into the first-order Lagrangian. This extra term, expanded as

$$G(a_i, p_i, \theta) = \sum_{n=0}^{\infty} \mathcal{G}^{(n)}(a_i, p_i, \theta) \tag{77}$$

where $\mathcal{G}^{(n)}(a_i, p_i, \theta)$ is a term of order n in θ, satisfies the following boundary condition:

$$G(a_i, p_i, \theta = 0) = \mathcal{G}^{(n=0)}(a_i, p_i, \theta = 0) = 0. \tag{78}$$

The symplectic variables are extended to contain also the WZ variable $\tilde{\xi}_{\tilde{\alpha}}^{(1)} = (\xi_\alpha^{(0)}, \eta, \theta)$ (with $\tilde{\alpha} = 1, 2, \ldots, 2N+2$) and the first-iterative symplectic potential becomes

$$\tilde{V}_{(n)}^{(1)}(a_i, p_i, \theta) = V^{(1)}(a_i, p_i) - \sum_{n=0}^{\infty} \mathcal{G}^{(n)}(a_i, p_i, \theta). \tag{79}$$

For $n = 0$, we have

$$\tilde{V}_{(n=0)}^{(1)}(a_i, p_i, \theta) = V^{(1)}(a_i, p_i). \tag{80}$$

Subsequently, we impose that the symplectic tensor ($f^{(1)}$) be a singular matrix with the corresponding zero-mode

$$\tilde{\nu}_{\tilde{\alpha}}^{(1)} = (\, \nu_\alpha^{(1)} \quad 1\,) \tag{81}$$

as the generator of gauge symmetry. Due to this, all correction terms $\mathcal{G}^{(n)}(a_i, p_i, \theta)$ in order of θ can be explicitly computed. Contracting the zero-mode ($\tilde{\nu}_{\tilde{\alpha}}^{(1)}$) with the gradient of potential $\tilde{V}_{(n)}^{(1)}(a_i, p_i, \eta, \theta)$ and imposing that no more constraints are generated, the following differential equation is obtained:

$$\begin{aligned}&\tilde{\nu}_{\tilde{\alpha}}^{(1)} \frac{\partial \tilde{V}_{(n)}^{(1)}(a_i, p_i, \theta)}{\partial \tilde{\xi}_{\tilde{\alpha}}^{(1)}} = 0 \\ &\nu_\alpha^{(1)} \frac{\partial V^{(1)}(a_i, p_i)}{\partial \xi_\alpha^{(1)}} - \sum_{n=0}^{\infty} \frac{\partial \mathcal{G}^{(n)}(a_i, p_i, \theta)}{\partial \theta} = 0\end{aligned} \tag{82}$$

which allows us to compute all correction terms in order of θ. For linear correction term, we have

$$\nu_{\alpha}^{(1)} \frac{\partial V_{(n=0)}^{(1)}(a_i, p_i)}{\partial \xi_{\alpha}^{(1)}} - \frac{\partial \mathcal{G}^{(n=1)}(a_i, p_i, \theta)}{\partial \theta} = 0 \tag{83}$$

while for the quadratic one:

$$\nu_{\alpha}^{(1)} \frac{\partial V_{(n=1)}^{(1)}(a_i, p_i, \theta)}{\partial \xi_{\alpha}^{(1)}} - \frac{\partial \mathcal{G}^{(n=2)}(a_i, p_i, \theta)}{\partial \theta} = 0. \tag{84}$$

From these equations, a recursive equation for $n \geqslant 1$ is proposed as

$$\nu_{\alpha}^{(1)} \frac{\partial V_{(n-1)}^{(1)}(a_i, p_i, \theta)}{\partial \xi_{\alpha}^{(1)}} - \frac{\partial \mathcal{G}^{(n)}(a_i, p_i, \theta)}{\partial \theta} = 0 \tag{85}$$

that allows us to compute each correction term in order of θ. This iterative process is repeated successively until equation (82) becomes identically null; consequently, the term $G(a_i, p_i, \theta)$ is obtained explicitly. At this stage, the gauge-invariant Hamiltonian, identified as being the symplectic potential, is obtained as

$$\tilde{\mathcal{H}}(a_i, p_i, \theta) = V_{(n)}^{(1)}(a_i, p_i, \theta) = V^{(1)}(a_i, p_i) + G(a_i, p_i, \theta) \tag{86}$$

and the zero-mode $\tilde{\nu}_{\tilde{\alpha}}^{(1)}$ is identified as being the generator of an infinitesimal gauge transformation:

$$\delta\tilde{\xi}_{\tilde{\alpha}} = \varepsilon \tilde{\nu}_{\tilde{\alpha}}^{(1)} \tag{87}$$

where ε is an infinitesimal time-dependent parameter.

Appendix 2. The gauge unfixing Hamiltonian formalism

The main idea of the unfixing gauge procedure is to consider half of the total second-class constraints as gauge fixing terms [20, 27] and the remaining as gauge generators of symmetry. To obtain a first-class Hamiltonian in a systematic way we follow closely the procedure described by Vytheeswaran in [20]. To start, we consider a system with two second-class constraints, ϕ_1 and ϕ_2, where the Poisson bracket is

$$C = \{\phi_1, \phi_2\}. \tag{88}$$

Using this relation and redefining the second-class constraints as

$$\begin{aligned} \xi &\equiv C^{-1}\phi_1 \\ \psi &\equiv \phi_2 \end{aligned} \tag{89}$$

we have

$$\{\xi, \psi\} = 1 + \{C^{-1}, \psi\}C\xi \tag{90}$$

so that ξ and ψ are canonically conjugate on the surface defined by $\xi = 0$.

The total Hamiltonian is

$$H = H_c + u_1\xi + u_2\psi. \tag{91}$$

To obtain the first-class system, we maintain only ξ as a constraint relation. At first, $\{\xi, H\} \neq 0$, i.e., ξ and H, in principle, do not satisfy a first-class algebra. Thus, the first-class Hamiltonian can be expressed by the formula given in [20], i.e.

$$\tilde{H} = H - \psi\{\xi, H\} + \frac{1}{2!}\psi^2\{\xi, \{\xi, H\}\} - \frac{1}{3!}\psi^3\{\xi, \{\xi, \{\xi, H\}\}\} + \cdots \tag{92}$$

which satisfies the first-class condition

$$\{\xi, \tilde{H}\} = 0. \tag{93}$$

The first-class Hamiltonian $\tilde{H}$ can be elegantly rewritten in a projection equation form, given by

$$\tilde{H} = PH \equiv: \exp^{-\psi\xi} : H \tag{94}$$

with ψ respecting the ordering rule, that is, it should come before the Poisson bracket. The procedure described above is an outline of a formalism that converts a second-class system into a first-class one without enlargement of the phase space.

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